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RESEARCH ARTICLE

Bayesian decision theory with action-dependent probabilities and risk attitudes

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Abstract This paper extends the work of Karni (Econ Theory 48:125–146, 2011) to allow for the possibility that decision makers' effect-dependent risk attitudes are also affected by their actions. This extension is essential for modeling decision situations in which actions have a monetary dimension that creates interaction between actions and wealth.

Keywords Bayesian decision theory · Subjective probabilities · Prior distributions · Beliefs · Constant utility bets

JEL Classification D80 · D81 · D82

1 Introduction

In this paper, I generalize the model of Karni (2011), allowing for action–bet interaction and, consequently, the possibility that the decision maker's risk attitudes may be affected not only by the effects but also by his actions.

Consider the following example which, in addition to motivating the extension, also lends concrete meaning to the abstract model described below. A homeowner must decide which measures should he take to protect his property against theft and fire (e.g., installing an alarm system, lights, fortified doors and windows, sprinklers, and a safety deposit box, having regular inspection of the electrical wiring). At the same time, the homeowner must also decide whether to take out a homeowner insurance policy, and if he decide to do so, what sort of coverage should the policy include. Before taking

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self-protecting actions and buying insurance, the homeowner may receive information concerning crimes and theft in the neighborhood in which he lives, which is pertinent to his decision. The homeowner is supposed to be able to design a contingent plan that specifying the measures of self-protection to be implemented and insurance policy to be taken out, contingent on the information available at the time this joint decision must be made.

Karni (2011) introduced a analytical framework that consists of a set, Θ, of effects, depicting physical phenomena on which the decision maker may place bets and which may or may not impact his well-being; a set, B, of such bets; a set, A, of actions, or initiatives, the decision maker can take in the belief that he can affect the likelihoods of ensuing effects; and a set of signals, $\bar{X}$, received before taking actions and choosing bets which may be relevant for his assessment of the likelihoods of the effects. In the example above, effects are potential losses due to theft or fire, actions depict the self-protection measures intended to reduce the magnitude and likelihoods of such losses. Bets correspond to alternative insurance policies and signals are crime reports.

In Karni (2011), the choice set, $\mathcal{I}$, consists of strategies for choosing actions and bets contingent on the signals received. Decision makers are characterized by a preference relation on $\mathcal{I}$ and are represented by

$$I \mapsto \sum_{x\in\bar{X}} \left[\sum_{\theta\in} \pi\left(\theta, x \mid a_{I(x)}\right) \left[u\left(b_{I(x)}(\theta), \theta\right) + v\left(a_{I(x)}\right)\right] \right], \tag{1}$$

where $a_{I(x)}$ and $b_{I(x)}$ are the action and bet assigned to the observation x by the strategy I; $\{u(\cdot, \theta)\}_{\theta\in}$ are effect-dependent utility functions on the monetary payoffs of the bets; v is the (dis)utility of actions; and $\{\pi(\cdot, \cdot \mid a)\}_{a\in A}$ is a unique family of action-dependent, joint, subjective probabilities distributions on $\times\, \bar{X}$ such that the prior distributions $\{\pi(\cdot \mid o, a)\}_{a\in A}$ and the posterior distributions $\{\pi(\cdot \mid x, a)\}_{a\in A}$ on are linked by Bayes rule and represent the decision maker's prior and posterior beliefs.

In the example above, actions are evaluated by their effectiveness in reducing the likelihood of property loss and their financial costs. The financial cost cannot be separated from the decision maker's wealth. This aspect of the problem cannot be satisfactorily handled by the model in which the (dis)utility of actions and the utility of wealth are additively separable. Extending the model to address this difficulty is the main objective of this paper. It is worth underscoring that the interaction among actions and other variables affecting the decision maker's well-being is not limited to financial considerations and may include inconvenience and effort. More formally, the objective of this paper is to develop a model in which representation (1) is replaced by the more general form

$$I \mapsto \sum_{x\in\bar{X}} \sum_{\theta\in} \pi\left(\theta, x \mid a_{I(x)}\right) u\left(a_{I(x)}, b_{I(x)}(\theta), \theta\right)), \tag{2}$$

where the utility functions $\{u(a, b(\theta), \theta))\}_{\theta\in}$ are not necessarily separately additive over actions and bets.

Attaining this objective requires two main changes to the original model: the axiom of independent betting preferences of Karni (2011) is weakened to include action-dependent betting preferences and a new concept—strings of constant utility bets—is introduced and incorporated into the analysis. The concept of strings of constant utility bets is a novel idea which, in addition to being essential for the problem at hand, represents a significant advance over the earlier model.

These modifications, however, do not alter the methodological approach that remains choice-based and Bayesian. The choice-based aspect maintains that a decision maker's choice among alternative strategies reflects his tastes for the ultimate outcomes and his beliefs regarding the likelihoods of the events in which these outcomes materialize. Consequently, the utility representing the decision maker's tastes and the probabilities representing his beliefs can be inferred from his choice behavior. The Bayesian aspect of the model is captured by the fact that new information affects the decision maker's posterior preferences, or choice behavior, solely through its effect on his beliefs, leaving the representation of his tastes intact, and that the posterior probabilities, representing the decision maker's posterior beliefs, are obtained by the updating the prior probabilities, representing his prior beliefs, using Bayes' rule.

Whether a decision maker's beliefs are a measurable cognitive phenomenon that can be quantified by the probabilities that figure in the subjective expected utility theory is debatable.[1] Nau (2011) takes the position that "It is… generally impossible to uniquely separate probabilities from utilities based on observations of the decision maker's preferences among bets (or any other concrete acts, for that matter)…" (Nau 2011, p. 440). Nau (1995, 2011) also claim that, insofar the theory of decision making under uncertainty is concern, the state-preference approach is preferable for its generality (it does not require the imposition of Savage (1954) sure-thing principle, or Savage's postulates P3 and P4 asserting that the preferences are relation are ordinarily and cardinally state-independent). He defines risk neutral probabilities from the decision makers marginal betting rates on events, which are the normalized products of subjective probabilities and marginal utilities. The approach taken in this paper and in Karni (2011) is that by extending the analytical framework, it is indeed possible to uniquely separate probabilities from utilities based on observations of the decision maker's preferences among strategies. Moreover, in Karni (2011a), I show how the probabilities on effects defined here induce unique subjective probabilities on an underlying state space.

Section 2 describes the analytical framework, the preference structure, and the main representation theorem. Concluding remarks appear in Sect. 3. The proof of the main result is given in the appendix.

2 The model

2.1 The analytical framework

Let be a finite set of *effects;* let A be a connected separable topological space, whose elements are referred to as *actions*; let X a finite set of *observations;* denote by o the

[1] See Karni (2011a) for a more detailed discussion and references.

event that no observation materializes and define $\bar{X} = X \cup \{o\}$.[2] A *bet* is a real-valued mapping on Θ, interpreted as monetary payoffs contingent on the realized effect. Let B denotes the set of all bets and assume that it is endowed with the $\mathbb{R}^{|\ \ |}$ topology. Denote by $b_{-\theta}r$ the bet obtained from $b \in B$ by replacing the θ-coordinate of b, $b(\theta)$, with r.

Informative and noninformative signals in the form of observation may be received by the decision maker before he chooses a bet and an action, and affect his choice. The decision maker is supposed to formulate a strategy specifying the action–bet pairs to be implemented contingent on the observations. Formally, a *strategy* is a function $I : \bar{X} \to A \times B$ whose interpretation is a set of instructions specifying, for each informational event an action–bet pair, $I(x)$, to be implemented if the informational event x obtains. Let $\mathcal{I}$ denotes the set of all strategies.

A decision maker is characterized by a preference relation $\succcurlyeq$ on $\mathcal{I}$. The strict preference relation, , and the indifference relation, $\sim$, are the asymmetric and symmetric parts of $\succcurlyeq$, respectively.

As usual, a consequence depicts those aspects of the decision problem that affect the decision maker's ex-post well-being. In this model, a *consequence* is a triplet (a, r, θ) representing, respectively, the action, the monetary payoff of the bet, and the effect. The set of all consequences is given by the Cartesian product $C = A \times \mathbb{R} \times\ $.

Denote by $I_{-x}(a, b)$ the strategy in which the x-coordinate of I, $I(x)$, is replaced by (a, b). The truncated strategy I_{-x} is referred to as a sub-strategy. For every given $x \in \bar{X}$, denote by $\succcurlyeq^x$, the induced preference relation on $A \times B$ defined by $(a, b) \succcurlyeq^x \left(a\ , b\ \right)$ if and only if $I_{-x}(a, b) \succcurlyeq I_{-x}\left(a\ , b\ \right)$. The induced strict preference relation, denoted by $\ ^x$, and the induced indifference relation, denoted by $\sim^x$, are the asymmetric and symmetric parts of $\succcurlyeq^x$, respectively.[3] The induced preference relation $\succcurlyeq^o$ is referred to as the *prior* preference relation; the preference relations $\succcurlyeq^x$, $x \in X$, are the *posterior* preference relations. An observation, x, is *essential* if $(a, b)\ \ ^x \left(a\ , b\ \right)$ for some (a, b), $\left(a\ , b\ \right) \in A \times B$. I assume throughout that all elements of $\bar{X}$ are essential.

For every $a \in A$ and $x \in \bar{X}$, define a binary relation $\succcurlyeq^x_a$ on B by: for all $b, b\ \in B$, $b \succcurlyeq^x_a b\ $ if and only if $(a, b) \succcurlyeq^x \left(a, b\ \right)$. The asymmetric and symmetric parts of $\succcurlyeq^x_a$ are denoted by $\ ^x_a$ and $\sim^x_a$, respectively.

An effect, θ, is said to be *nonnull given the observation–action pair* (x, a) if $(b_{-\theta}r)\ \ ^x_a \left(b_{-\theta}r\ \right)$, for some $b \in B$ and $r, r\ \in \mathbb{R}$; it is *null given the observation–action pair* (x, a) otherwise. Given a preference relation, $\succcurlyeq$, denote by $\ (a, x)$, the subset of effects that are nonnull given the observation–action pair (x, a). Assume that $\ (a, o) = \Theta$, for all $a \in A$.

2.2 The preference structure

Consider the following axioms depicting the structure of a preference relation $\succcurlyeq$ on $\mathcal{I}$. With slight variations in axioms (A.4), (A.6), and (A.7), all the axioms below were

[2] The interpretation of these terms is as in the introduction.

[3] For preference relations satisfaying (A.1)–(A.3) below, these relations are well-defined. In particular, they are independent of I.

introduced and their meaning discussed, in Karni (2011). I therefore refrain from further elaboration here.

(A.1) **(Weak order)** $\succcurlyeq$ *is a complete and transitive binary relation.*

A topology on $\mathcal{I}$ is needed to define continuity of the preference relation $\succcurlyeq$. Recall that $\mathcal{I} = (A \times B)^{\bar{X}}$, and let $\mathcal{I}$ be endowed with the product topology.[4]

(A.2) **(Continuity)** *For all* $I \in \mathcal{I}$, *the sets* $\{I \ \in \mathcal{I} \mid I \ \succcurlyeq I\}$ *and* $\{I \ \in \mathcal{I} \mid I \succcurlyeq I \ \}$ *are closed.*

The next axiom, coordinate independence, is analogous to but weaker than Savage (1954) sure-thing principle.[5] Like the sure-thing principle, it requires that strategies be compared independently of the aspects (coordinates) on which they agree.

(A.3) **(Coordinate independence)** *For all* $x \in \bar{X}$, $I, I \ \in \mathcal{I}$, *and* $(a, b), \left(a \ , b \ \right) \in A \times B$, $I_{-x}(a, b) \succcurlyeq I_{-x}(a, b)$ *if and only if* $I_{-x}\left(a \ , b \ \right) \succcurlyeq I_{-x}\left(a \ , b \ \right)$.

The next axiom requires that the "intensity of preferences" for monetary payoffs contingent on any given effect be independent of the observation. It is a weakening of axiom (A.4) in Karni (2011), which required, in addition, that the effect-contingent "intensity of preferences" for monetary payoffs be independent of the actions. To grasp the meaning of this axiom, note that if the payoffs were roulette lotteries a la Anscombe and Aumann (1963), then the condition would amount to the requirement that, given any action and effect, the ranking of (roulette) lotteries contingent on that action and effect be observation-independent. This would allow the decision maker's risk attitudes to be action and effect-dependent but observation-independent. To avoid invoking the notion of probabilities as an primitive, it is necessary to measure the intensity of preferences in some other way.[6] To accomplish this, I extend the trade-off method of Wakker (1987). In particular, fix an action, a, an effect, θ, and an observation, x, and suppose that $(b_{-\theta} r) \sim_a^x \left(b_{-\theta} r \ \right)$ and $\left(b_{-\theta} r \ \right) \sim_a^x \left(b_{-\theta} r \ \right)$. These indifferences have the interpretation that, given an action, and effect and an observation, the "intensity of preferences" between r and r is the same as that between r and r , and they are both *measured* by the difference between the sub-bets $b_{-\theta}$ and $b_{-\theta}$. Now, holding the action and bet the same, consider the issue of "intensity of preferences" under another observation x (instead of x). The axiom requires that the "intensity of preferences" between r and r remains the same as that between r and r . In other words, if the intensity of preferences between r and r is measured by the sub-bets $b_{-\theta}$ and $b_{-\theta}$, (that is, let $\left(b_{-\theta} r \ \right) \sim_a^x \left(b_{-\theta} r\right)$), then that between r and r must be the same, namely $\left(b_{-\theta} r \ \right) \sim_a^x \left(b_{-\theta} r \ \right)$. Formally,

(A.4) **(Observation-independent action-betting preferences)** *For all* $x, x \ \in \bar{X}, b, b \ , b \ , b \ \in B, \theta \in \ (x) \cap \ \left(x \ \right)$, *and* $r, r \ , r \ , r \ \in \mathbb{R}$, *if*

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[4] Recall that A is a topological space and assume that B is endowed with the $\mathbb{R}^n$ topology. Then the topology on $\mathcal{I}$ is the product topology on the Cartesian product $(A \times B)^{|\bar{X}|}$.

[5] See Wakker (1989) for details.

[6] In this sense, following Savage (1954), I pursue the purely subjective approach avoiding the use of probabilities as a primitives. The cardinality of the utility functions needs to be imposed by other means.

$(a, b_{-\theta}r) \succcurlyeq^x \left(a, b_{-\theta}r\right)$, $\left(a, b_{-\theta}r\ \right) \succcurlyeq^x \left(a, b_{-\theta}r\ \ \right)$, *and* $\left(a, b_{-\theta}r\right) \succcurlyeq^x$ $\left(a, b_{-\theta}r\right)$ *then* $\left(a, b_{-\theta}r\ \right) \succcurlyeq^x \left(a, b_{-\theta}r\ \ \right)$.

To link the decision maker's prior and posterior probabilities, the next axiom asserts that, in and of itself, information is worthless. To state this axiom, let $I^{-o}(a, b)$ denote the strategy that assigns the action–bet pair (a, b) to every observation other than o (that is, $I^{-o}(a, b)$ is a strategy such that $I(x) = (a, b)$ for all $x \in X$). The implication of adopting this strategy is that the action–bet pair to be implemented is the same, regardless the information that may be acquired. In other words, given this strategy, information is useless. The axiom requires that, given an action, the preferences on bets when new information may not be used to select the bet be the same as the preference relation conditional on no new information.

(A.5) (**Belief consistency**) *For every* $a \in A$, $I \in \mathcal{I}$ *and* $b, b \in B$, $I_{-o}(a, b) \sim I_{-o}\left(a, b\right)$ *if and only if* $I^{-o}(a, b) \sim I^{-o}\left(a, b\right)$.

2.3 Strings of constant utility bets

Bets whose payoffs offset the direct impact of the effects are *constant utility bets*. Because of the weakening of (A.4), unlike in Karni (2011), in this paper, the constant utility bets are not independent of the actions. This requires a modification of the analysis and a new concept, dubbed *strings of constant utility bets*.

To grasp intuition underlying the formal definition of strings of constant utility bets, it is convenient to consider first the special case in which the valuation of the bets is independent of the actions. Suppose that the bet b satisfies the following conditions $I_{-x}\left(a, b\right) \sim I_{-x}\left(a\ , b\right)$ and $I_{-x}\left(a\ \ , b\right) \sim I_{-x}\left(a\ \ \ , b\right)$ for some observation x, strategies I, I , and actions $a, a\ , a\ \ , a\ \ \ $. Then, given b and x, the indifference $I_{-x}\left(a, b\right) \sim I_{-x}\left(a\ , b\right)$ depicts compensating variations between the sub-strategy I_{-x} and that action a , and the sub-strategy I_{-x} and the action a . Similarly, the indifference $I_{-x}\left(a\ \ , b\right) \sim I_{-x}\left(a\ \ \ , b\right)$ depicts compensating variations between the sub-strategy I_{-x} and that action a , and the sub-strategy I_{-x} and the action a . Hence, the difference between sub-strategies I_{-x} and I_{-x} "measures" the difference in the intensity of preference between a and a and also that between a and a .[7]

Recall that the choice of action affects the decision maker's well-being directly, (the disutility of action) and indirectly, through its effect on the probabilities of the alternative effects. For the second effect to be manifested, the utility must display some variation across effects. Constant utility bets, and only constant utility bets, are distinguished by the lack of such variations. Hence, the second effect is neutralized if and only if the bet under consideration is constant utility. For such bets, solely the direct impact of the action is manifested. The definition of constant utility bets rules out distinct affine transformations of the utility functions across mutually exclusive effects. Moreover, because the impact of the observations on the decision maker's well-being

[7] In this case, the intensity of preferences between a and a is, in fact, the same as that between a and a .

is through the probabilities, the definition of constant utility bets requires that the intensity of preferences between any two actions be independent of the observations.

With this in mind, consider the observation x and suppose that $I_{-x}\left(a, b\right) \sim I_{-x}\left(a, b\right)$. These compensating variations imply that, given x, the measure of the intensity of preference between a and a is the difference in the sub-strategies I_{-x} and I_{-x}. The definition of constant utility bets requires that the measure of the intensity of the preference between a and a, *being observation-independent*, is also given by the difference in the sub-strategies I_{-x} and I_{-x}.

The same intuition applies to the more general case in which the impacts of the actions and bets on the decision maker's well-being are not separable. In this instance, however, what constitute constant utility bets depend on the actions. Consequently, the difference between the sub-strategies I_{-x} and I_{-x} measures the intensity of preferences between the actions taking into account that the associated constant utility of the bets varies with the actions. Nevertheless, the crucial point remains the same, namely when the intensity of preference between the actions and the corresponding bets is *independent of the observations,* the indirect impact of the actions and that of the observations must has been neutralized, indicating that the corresponding bets are constant utility. Formally,

Definition 1 A mapping $\bar{b} : A \to B$ is **a string of constant utility bets according to** $\succcurlyeq$ if, for all $I, I, I, I \in \mathcal{I}, a, a, a, a \in A$ and $x, x \in \bar{X}$, $I_{-x}\left(a, \bar{b}(a)\right) \sim I_{-x}\left(a, \bar{b}\left(a\right)\right), I_{-x}\left(a, \bar{b}\left(a\right)\right) \sim I_{-x}\left(a, \bar{b}\left(a\right)\right)$ and $I_{-x}\left(a, \bar{b}(a)\right) \sim I_{-x}\left(a, \bar{b}\left(a\right)\right)$ imply $I_{-x}\left(a, \bar{b}\left(a\right)\right) \sim I_{-x}\left(a, \bar{b}\left(a\right)\right)$ and $\cap_{x \in X}\{b \in B \mid b \sim_a^x \bar{b}(a)\} = \{\bar{b}(a)\}$, for all $a \in A$.

To render the definition meaningful, it is assumed that, given a string of constant utility bets $\bar{b}$, for all $a, a, a, a \in A$ and $x, x \in \bar{X}$ there are $I, I, I, I \in \mathcal{I}$ such that the indifferences $I_{-x}\left(a, \bar{b}(a)\right) \sim I_{-x}\left(a, \bar{b}\left(a\right)\right), I_{-x}\left(a, \bar{b}\left(a\right)\right) \sim I_{-x}\left(a, \bar{b}\left(a\right)\right)$ and $I_{-x}\left(a, \bar{b}(a)\right) \sim I_{-x}\left(a, \bar{b}\left(a\right)\right)$ hold.

Let $\mathcal{B}(\succcurlyeq)$ denotes the set of all strings of constant utility bets according to $\succcurlyeq$.

Definition 2 The set $\mathcal{B}(\succcurlyeq)$ is **inclusive** if there is $\bar{b} \in \mathcal{B}(\succcurlyeq)$ such that $(a, b) \sim^x \left(a, \bar{b}(a)\right)$, for every $x \in X$ and $(a, b) \in A \times B$.

If, for some actions, there exists no monetary compensation for the impact of the effects (that is, the ranges of the utility of the monetary payoffs across effects do not overlap), then, for that action, there is no constant utility bet and $\mathcal{B}(\succcurlyeq)$ is empty. Here, I am concerned with the case in which $\mathcal{B}(\succcurlyeq)$ is inclusive, and thus nonempty. Constant utility bets may be thought of as providing a hedge against the utility variations due to the realization of distinct effects.

In the special case, $I = I$ and $I = I$, Definition 1 implies that $\left(a, \bar{b}(a)\right) \sim^x \left(a, \bar{b}\left(a\right)\right)$ for all $x \in \bar{X}$. Anticipating the main result, this means that $\left(a, \bar{b}(a)\right)$ and $\left(a, \bar{b}\left(a\right)\right)$ correspond to the same expected utility, regardless of the observation.[8] This special case pertains, naturally, to actions identified with monetary expenses that are perfect substitutes for the payoffs of the bets. In general, however, it is possible that

[8] I thank Jacques Drèze for calling my attention to this special case.

there are no feasible monetary compensation for the disutility associated with some actions. In such a case, the expected utilities associated with $\left(a, \bar{b}(a)\right)$ and $\left(a\ , \bar{b}\left(a\ \right)\right)$ are distinct, and the difference between them is "measured" by the utility difference between the sub-strategies I_{-x} and I_{-x} as well as that between I_{-x} and I_{-x}.

The next two axioms are rather straightforward. The first requires that the trade-offs between the actions and the sub-strategies that figure in Definition 1 be independent of the constant utility bets. Formally,

(A.6) (**Trade-off independence**) For all $I, I\ \in \mathcal{I}, x \in \bar{X}, a, a\ \in A$ and $\bar{b}, \bar{b}\ \in \mathcal{B}(\succcurlyeq), I_{-x}\left(a, \bar{b}(a)\right) \succcurlyeq I_{-x}\left(a\ , \bar{b}\left(a\ \right)\right)$ if and only if $I_{-x}\left(a, \bar{b}\ (a)\right) \succcurlyeq I_{-x}\left(a\ , \bar{b}\ \left(a\ \right)\right)$.

Finally, it is also required that the direct effect (that is, the cost) of actions, measured by the preferential difference between any two strings of constant utility bets, $\bar{b}, \bar{b}\ \in \mathcal{B}(\succcurlyeq)$, be independent of observation. Formally,

(A.7) (**Conditional monotonicity**) For all $\bar{b}, \bar{b}\ \in \mathcal{B}(\succcurlyeq), x, x\ \in \bar{X}$, and $a, a\ \in A$, $\left(a, \bar{b}(a)\right) \succcurlyeq^{x} \left(a\ , \bar{b}\ \left(a\ \right)\right)$ if and only if $\left(a, \bar{b}(a)\right) \succcurlyeq^{x}\ \left(a\ , \bar{b}\ \left(a\ \right)\right)$.

2.4 Representation

The next theorem generalizes Theorem 2 of Karni (2011) by permitting interaction between actions and bets. Consequently, the effect-dependent utility functions are not necessarily separately additive in actions and bets. To simplify the statement of the results that follow, I let $I(x) = \left(a_{I(x)}, b_{I(x)}\right)$.

Theorem 3 *Let $\succcurlyeq$ be a preference relation on $\mathcal{I}$ and suppose that $\mathcal{B}(\succcurlyeq)$ is inclusive, then*

(a) *The following conditions are equivalent:*
  (i) *$\succcurlyeq$ satisfies (A.1)–(A.7).*
  (ii) *there exist a continuous, real-valued function u on $A \times \mathbb{R} \times$, and a family of joint probability measures $\{\pi(\cdot, \cdot \mid a)\}_{a \in A}$ on $\bar{X} \times$ such that $\succcurlyeq$ on I is represented by*

$$I \mapsto \sum_{x \in \bar{X}} \mu(x) \sum_{\theta \in} u\left(a_{I(x)}, b_{I(x)}(\theta), \theta\right)) \pi\left(\theta \mid x, a_{I(x)}\right) \tag{3}$$

where $\mu(x) = \ _{\theta \in}\ \pi(x, \theta \mid a)$ for all $x \in \bar{X}$ is independent of a and, for each $a \in A, \pi(\theta \mid x, a) = \pi(x, \theta \mid a) / \mu(x)$ for all $x \in X$, and $\pi(\theta \mid o, a) = \frac{1}{1-\mu(o)}\ _{x \in X} \pi(x, \theta \mid a)$.

(b) *The function u is unique up to a positive affine transformation and, for each $a \in A, \pi(\cdot, \cdot \mid a)$ is unique.*

(c) *For every $\bar{b} \in B(\succcurlyeq)$ and $a \in A$, $u\left(a, \bar{b}(a)(\theta), \theta\right) = u\left(a, \bar{b}(a)\left(\theta\ \right), \theta\ \right)$ for all $\theta, \theta\ \in$.*

Notice that, although the joint probability distributions $\pi(\cdot, \cdot \mid a), a \in A$ depend on the actions, the distribution μ is independent of a. This is consistent with the formulation of the decision problem according to which the choice of actions is contingent on the observations. In other words, if new information arrives, it precedes the choice of action. Hence, the dependence of the joint probability distributions $\pi(\cdot,\cdot \mid a)$ on a captures solely the decision maker's beliefs about his ability to influence the likelihood of the effects by his choice of action.[9]

A special case of Theorem 3 obtains when actions are monetary expenditures (that is, when $A = \mathbb{R}_-$). For instance, when considering installing sprinklers to reduce the loss in case of a fire, it is natural to assume that the utility impact of this action depends solely on the money spent. Hence, $u(a, b(\theta), \theta)) = u(a + b(\theta), \theta)), \theta \in$. In general, actions affect the preference directly, through their associated disutility and the possible associated "wealth effect" on the decision maker's attitudes toward the risk represented by the bets, and indirectly, through their impact on the probabilities of the effects. To isolate the "utility impact," it is necessary to confine attention to strings of constant utility bets. The idea that, insofar as the utility is concerned, actions and bets are perfect substitutes is captured by the following axiom:

(A.8) (**Substitution**) For all $\bar{b} \in \mathcal{B}(\succcurlyeq), I \in \mathcal{I}, x \in \bar{X}$ and $a \in \mathbb{R}_-, z \in \mathbb{R}$, $I_{-x}\left(a, \bar{b}(a)\right) \sim I_{-x}\left(a - z, \bar{b}(a) + z\right)$.

By Theorem 3 and axiom (A.8), for every $\bar{b} \in \mathcal{B}(\succcurlyeq)$ and $a \in A$, $u\left(a, \bar{b}(a)(\theta), \theta\right) = u\left(a - z, \bar{b}(a)(\theta) + z, \theta\right)$ for all $\theta \in$. Hence, with slight abuse of notations, $u\left(a, \bar{b}(a)(\theta), \theta\right) = u\left(a + \bar{b}(a)(\theta), \theta\right)$ for $\theta \in$. This implies the following:

Corollary 4 *Let $A = \mathbb{R}_-$ and $\succcurlyeq$ be a preference relation on $\mathcal{I}$ and suppose that $\mathcal{B}(\succcurlyeq)$ is inclusive. Then, $\succcurlyeq$ satisfies (A.1)–(A.8) if and only if there exist a continuous, real-valued function u on $A \times \mathbb{R} \times$, and a family of joint probability measures $\{\pi(\cdot,\cdot \mid a)\}_{a\in A}$ on $\bar{X} \times$ such that $\succcurlyeq$ on $\mathcal{I}$ is represented by*

$$I \mapsto \sum_{x\in\bar{X}} \mu(x) \sum_{\theta\in} u\left(a_{I(x)} + b_{I(x)}(\theta), \theta\right))\pi\left(\theta \mid x, a_{I(x)}\right) \tag{4}$$

where $\mu(x) = {}_{\theta\in}\ \pi(x, \theta \mid a)$ for all $x \in \bar{X}$ is independent of a and, for each $a \in A, \pi(\theta \mid x, a) = \pi(x, \theta \mid a)/\mu(x)$ for all $x \in X$, and $\pi(\theta \mid o, a) = \frac{1}{1-\mu(o)}\ {}_{x\in X}\ \pi(x, \theta \mid a)$. Moreover, u is unique up to positive affine transformation, for each $a \in A \pi(\cdot,\cdot \mid a)$ is unique and, for every $\bar{b} \in \mathcal{B}(\succcurlyeq)$ and $a \in A$, $u\left(a + \bar{b}(a)(\theta), \theta\right) = u\left(a + \bar{b}(a)\left(\theta\ \right), \theta\ \right)$ for all $\theta, \theta \in$.

3 Concluding remarks

This paper presents a model of Bayesian decision making under uncertainty that allows for effect-dependent and action-dependent risk attitudes. In addition to being applicable to the analysis of decisions in which the effect interacts with the decision maker's

[9] If an action-bet pair are already "in effect" when new information arrives, they constitute a default course of action. In such instance, the interpretation of the decision at hand is possible choice of new action and bet. For example, a modification, upon learning the results of medical checkup, of a diet regiment coupled with a possible change of life insurance policy.

wealth, such as the choice of life and health insurance, it is also useful for the analysis of decisions in which actions affect decision maker's risk attitudes. This aspect is crucial in situations in which the actions themselves involve monetary expenses. Special instances of the theory presented here, including representations of preference relations displaying effect-independence preferences, may be obtained in a straightforward manner following Karni (2011).

The uncertainty modeled of this paper resolves in two stages. In the first stage, an observation obtains, which partially resolves the uncertainty, followed by an interim stage, in which an action and a bet are chosen. In the second stage, an effect is realized, thereby resolving the remaining uncertainty, and the payoff of the bet is effectuated. The gradual resolution of uncertainty bears some resemblance to temporal resolution of uncertainty modeled and analyzed by Kreps and Porteuse (1978, 1979). Despite the similarities, however, the models are fundamentally different. First, the theory of Kreps and Porteus concerns risk rather than uncertainty. In other words, taking the set of lotteries over basic sequences of dated outcomes (e.g., consumption streams) as primitive, Kreps and Porteus' main concern is the representation of induced preferences on temporal lotteries whose support is a set of sequences of dated payoffs and actions (e.g., income streams and saving decisions) as temporal von Neumann–Morgenstern preferences. In particular, they are interested in the representation of distinct attitudes toward the timing of resolution of uncertainty, the modeling of which requires that the domain of the preference relation be dated compound lotteries. By contrast, the main concern of this work is decision making under uncertainty and, in particular, the development of an analytical framework and axiomatic subjective expected utility models supporting the existence and uniqueness of a set of action-dependent subjective probabilities that represent decision makers' beliefs and are updated according to Bayes' rule. Naturally, these differences in objectives and analytical frameworks project upon the preference structures of the corresponding models, their representations, and their applications. In particular, the representation of temporal von Neumann–Morgenstern preferences has a recursive expected utility structure and its main applications are in dynamic choice problem. By contrast, the preference relations modeled in this work have subjective expected utility representation and is applicable to the theory of moral hazard, in which actions affect the likelihoods of the outcomes.[10] Moreover, unlike in the theory of Kreps and Porteus, the gradual resolution of uncertainty in this model is manifested by the updating of the prior preference relation over actions and bets, in view of new information, according to Bayes' rule. Because it does not involve choice among temporal lotteries (that is, lotteries encoding the time resolution of uncertainty), this model is silent on decision makers' attitudes toward the timing resolution of uncertainty. Because for any given action–observation pair, the conditional preference relation over bets is linear in the probabilities, the approach taken here may seem to correspond to what Kreps and Porteus call atemporal von Neumann–Morgenstern preferences. However, in view of the fact, this model does not allow the expression of attitudes toward the timing of the resolution of uncertainty, this interpretation is misleading. Put differently, this model is neither a special case of Kreps

[10] This model can be regarded as a subjective version of Mirrlees' (1974, 1976) model in which the agent's choice of action affects the likely realizations of outcomes.

and Porteus-induced preference approach, nor is the atemporal-induced preferences, a special case of this model. Second, the analogue in this work of temporal lotteries is strategies, and the analogue of the first-period payoff in Kreps and Porteus is an observation. However, while the first-period payoffs in Kreps and Porteus (1979) have informational content as well as material implications that affect the decision maker well-being directly, in this model, observations have information content affecting the decision maker's beliefs, but are devoid of material implications. Third, the actions in the model of Kreps and Porteus affect the decision maker's well-being directly, allowing the decision maker to trade-off the payoffs across periods. In this work, the actions affect the decision maker's well-being both directly, as they impose a cost, and indirectly by allowing the decision maker to influence the likely realization of the alternative effects. The model is silent on the issue of intertemporal allocation. Finally, because the induced preferences in Kreps and Porteus (1979) are defined on distributions over objects whose design requires interim optimization, except under stringent conditions, they do not display the "linearity in probabilities" property of expected utility theory.[11] This issue does not arise in this work since the decision maker choice involves strategies (that is, optimal action–bet pairs contingent on the observation). The model does not include induced preferences on distributions over strategies.

Appendix

For expository convenience, I write $\mathcal{B}$ instead of $\mathcal{B}(\succcurlyeq)$.

Proof of Theorem 3

(a) (Sufficiency) Assume that $\succcurlyeq$ on $\mathcal{I}$ satisfies (A.1)–(A.7) and $\mathcal{B}$ is inclusive. Let $\mathcal{I}$ be endowed with the product topology and suppose that $|\bar{X}| \geq 3$.[12]

By Wakker (1989) Theorem III.4.1, $\succcurlyeq$ satisfies (A.1)–(A.3) if and only if there exists an array of real-valued functions $\{w(\cdot,\cdot,x) \mid x \in \bar{X}\}$ on $A \times B$ such that $\succcurlyeq$ is represented by

$$I \mapsto \sum_{x \in \bar{X}} w\left(a_{I(x)}, b_{I(x)}, x\right), \tag{5}$$

where $w(.,.,x)$, $x \in \bar{X}$ are jointly cardinal, continuous, real-valued functions.[13]

[11] See Machina (1984) for a discussion and non-expected utility modeling of induced preferences.

[12] To simplify the exposition I state the theorem for the case in which $\bar{X}$ contains at least three essential coordinates. Additive representation when there are only two essential coordinates requires the imposition of the hexagon condition (see Wakker 1989, Theorem III.4.1).

[13] An array of real-valued functions $(v_s)_{s\in S}$ is said to be a *jointly cardinal additive representation* of a binary relation on a product set $D = {}_{s\in S} D_s$ if, for all $d, d \in D$, $d \quad d$ if and only if ${}_{s\in S} v_s(d_s) \geq {}_{s\in S} v_s\left(d_s\right)$, and the class of all functions that constitute an additive representation of consists of those arrays of functions, $\left(\hat{v}_s\right)_{s\in S}$, for which $\hat{v}_s = \lambda v_s + \zeta_s$, $\lambda > 0$ for all $s \in S$. The representation is continuous if the functions v_s, $s \in S$ are continuous.

Since $\succcurlyeq$ satisfies (A.4), Lemmas 5 and 6 in Karni (2006) and Theorem III.4.1 in Wakker (1989) imply that, for every $x \in \bar{X}$ and $a \in A$ such that $\quad (a, x)$ contains at least two effects, there exists array of functions $\{v_x (a, \cdot; \theta) : \mathbb{R} \to \mathbb{R} \mid \theta \in \quad \}$ constituting a jointly cardinal, continuous, additive, representation of $\succcurlyeq_a^x$ on B. Moreover, by the proof of Lemma 6 in Karni (2006), $\succcurlyeq$ satisfies (A.1)–(A.4) if and only if, for every $x, x \;\in \bar{X}$ and $a \in A$ satisfying $\quad (a, x) \cap \quad \left(a, x\;\right) = \varnothing$ and $\theta \in \quad (a, x) \cap \quad \left(a, x\;\right)$, there exists $\beta_{(x\;,x,a,\theta)} > 0$ and $\alpha_{(x\;,x,a,\theta)}$ such that $v_x \;(a, \cdot, \theta) = \beta_{(x\;,x,a,\theta)} v_x (a, \cdot, \theta) + \alpha_{(x\;,x,a,\theta)}$.[14]

Define $u (a, \cdot, \theta) = v_o (a, \cdot, \theta)$, $\lambda (a, x; \theta) = \beta_{(x,o,a,\theta)}$ and $\alpha (a, x, \theta) = \alpha_{(x,o,a,\theta)}$ for all $a \in A, x \in \bar{X}$, and $\theta \in \quad$. For every given $x \in \bar{X}$ and $a \in A$, $w (a, b, x)$ represents $\succcurlyeq_a^x$ on B. Hence,

$$w (a, b, x) = H \left(\sum_{\theta \in \quad} (\lambda (a, x, \theta) u (a, b (\theta) ; \theta) + \alpha (a, x, \theta)), a, x \right), \tag{6}$$

where H is a continuous, increasing function.

Consider next the restriction of $\succcurlyeq$ to $L := \{\left(a, \bar{b} (a)\right) \mid a \in A, \bar{b} \in \mathcal{B}\}^{\bar{X}}$.

Lemma 5 *There exists functions $U : L \to \mathbb{R}$, $\xi : \bar{X} \to \mathbb{R}_{++}$, and $\zeta : \bar{X} \to \mathbb{R}$ such that, for all $\left(a, \bar{b}, x\right) \in A \times \mathcal{B} \times \bar{X}$,*

$$w \left(a, \bar{b} (a), x\right) = \xi (x) U \left(a, \bar{b} (a)\right) + \zeta (x), \xi (x) > 0. \tag{7}$$

Proof Let $I, I\;, I\;\;, I\;\;\; \in \mathcal{I}, a, a\;, a\;\;, a\;\;\; \in A$ and $\bar{b} (a), \bar{b} \left(a\;\right), \bar{b} \left(a\;\;\right), \bar{b} \left(a\;\;\;\right)$ be as in Definition 1. Then, for all $x, x\; \in \bar{X}$, $I_{-x} \left(a, \bar{b} (a)\right) \sim I_{-x}\; \left(a\;, \bar{b} \left(a\;\right)\right)$, $I_{-x}\;\; \left(a\;\;, \bar{b} \left(a\;\;\right)\right) \sim I_{-x}\;\;\; \left(a\;\;\;, \bar{b} \left(a\;\;\;\right)\right)$, $I_{-x\;} \left(a, \bar{b} (a)\right) \sim I_{-x\;}\; \left(a\;, \bar{b} \left(a\;\right)\right)$ and $I_{-x\;}\;\; \left(a\;\;, \bar{b} \left(a\;\;\right)\right) \sim I_{-x\;}\;\;\; \left(a\;\;\;, \bar{b} \left(a\;\;\;\right)\right)$. By the representation (5), $I_{-x} \left(a, \bar{b} (a)\right) \sim I_{-x}\; \left(a\;, \bar{b} \left(a\;\right)\right)$ implies that

$$\sum_{y \in \bar{X} - \{x\}} w \left(a_{I(y)}, b_{I(y)}, y\right) + w \left(a, \bar{b} (a), x\right) = \sum_{y \in \bar{X} - \{x\}} w \left(a_{I\;(y)}, b_{I\;(y)}, y\right) + w \left(a\;, \bar{b} \left(a\;\right), x\right). \tag{8}$$

Similarly, $I_{-x}\;\; \left(a\;\;, \bar{b} \left(a\;\;\right)\right) \sim I_{-x}\;\;\; \left(a\;\;\;, \bar{b} \left(a\;\;\;\right)\right)$ implies that

$$\sum_{y \in \bar{X} - \{x\}} w \left(a_{I(y)}, b_{I(y)}, y\right) + w \left(a\;\;, \bar{b} \left(a\;\;\right), x\right) = \sum_{y \in \bar{X} - \{x\}} w \left(a_{I\;(y)}, b_{I\;(y)}, y\right) + w \left(a\;\;\;, \bar{b} \left(a\;\;\;\right), x\right), \tag{9}$$

14 By definition, for all (a, x) and θ, $\beta_{(x,x,a,\theta)} = 1$ and $\alpha_{(x,x,a,\theta)} = 0$.

$I_{-x}\left(a, \bar{b}(a)\right) \sim I_{-x}\left(a\ , \bar{b}\left(a\ \right)\right)$ implies that

$$\sum_{y \in \bar{X}-\{x\ \}} w\left(a_{I\ (y)}, b_{I\ (y)}, y\right)+w\left(a, \bar{b}(a), x\ \right)=\sum_{y \in \bar{X}-\{x\ \}} w\left(a_{I\ \ (y)}, b_{I\ \ (y)}, y\right) +w\left(a\ , \bar{b}\left(a\ \right), x\ \right), \tag{10}$$

and $I_{-x}\left(a\ , \bar{b}\left(a\ \right)\right) \sim I_{-x}\left(a\ \ , \bar{b}\left(a\ \ \right)\right)$ implies that

$$\sum_{y \in \bar{X}-\{x\ \}} w\left(a_{I\ (y)}, b_{I\ (y)}, y\right)+w\left(a\ , \bar{b}\left(a\ \right), x\ \right)=\sum_{y \in \bar{X}-\{x\ \}} w\left(a_{I\ \ (y)}, b_{I\ \ (y)}, y\right) +w\left(a\ \ , \bar{b}\left(a\ \ \right), x\ \right). \tag{11}$$

But (8) and (9) imply that

$$w\left(a, \bar{b}(a), x\right)-w\left(a\ , \bar{b}\left(a\ \right), x\right)=w\left(a\ , \bar{b}\left(a\ \right), x\right)-w\left(a\ \ , \bar{b}\left(a\ \ \right), x\right). \tag{12}$$

and (10) and (11) imply that

$$w\left(a, \bar{b}(a), x\ \right)-w\left(a\ , \bar{b}\left(a\ \right), x\ \right)=w\left(a\ , \bar{b}\left(a\ \right), x\ \right)-w\left(a\ \ , \bar{b}\left(a\ \ \right), x\ \right). \tag{13}$$

Define a function $\phi_{(x,x\ ,\bar{b})}$ as follows: $w\left(\cdot, \bar{b}(\cdot), x\right)=\phi_{(x,x\ ,\bar{b})} \circ w\left(\cdot, \bar{b}(\cdot), x\ \right)$. Then, $\phi_{(x,x\ ,\bar{b})}$ is continuous. Axiom (A.7) implies that $\phi_{(x,x\ ,\bar{b})}$ is monotonic increasing. Moreover, Eqs. (12) and (13) in conjunction with Lemma 4.4 in Wakker (1987) imply that $\phi_{(x,x\ ,\bar{b})}$ is affine.

Let $\beta_{(x,o,\bar{b})}>0$ and $\delta_{(x,o,\bar{b})}:=\zeta(x)$ denote, respectively, the multiplicative and additive coefficients corresponding to $\phi_{(x,o,\bar{b})}$, where the inequality follows from the monotonicity of $\phi_{(x,o,\bar{b})}$. Observe that $I_{-o}\left(a, \bar{b}(a)\right) \sim I_{-o}\left(a\ , \bar{b}\left(a\ \right)\right)$ and $I_{-o}\left(a, \bar{b}\ (a)\right) \sim I_{-o}\left(a\ , \bar{b}\ \left(a\ \right)\right)$ in conjunction with axiom (A.6) imply that

$$\beta_{(x,o,\bar{b})}\left[w\left(a, \bar{b}(a), o\right)-w\left(a\ , \bar{b}\left(a\ \right), o\right)\right]=\beta_{(x,o,\bar{b}\ )}\left[w\left(\cdot, \bar{b}\ (a), o\right) -w\left(a\ , \bar{b}\ \left(a\ \right), o\right)\right] \tag{14}$$

for all $\bar{b}, \bar{b}\ \in \mathcal{B}$. Thus, for all $x \in \bar{X}$ and $\bar{b}, \bar{b}\ \in \mathcal{B}$, $\beta_{(x,o,\bar{b})}=\beta_{(x,o,\bar{b}\ )}:=\xi(x)>0$.

Let $a, a\ \in A$ and $\bar{b}, \bar{b}\ \in \mathcal{B}$ satisfy $\left(a, \bar{b}(a)\right) \sim^{o}\left(a\ , \bar{b}\ \left(a\ \right)\right)$. By axiom (A.7) $\left(a, \bar{b}(a)\right) \sim^{x}\left(a\ , \bar{b}\ \left(a\ \right)\right)$ if and only if $\left(a, \bar{b}(a)\right) \sim^{o}\left(a\ , \bar{b}\ \left(a\ \right)\right)$. By the representation, this equivalence implies that

$$w\left(a, \bar{b}(a), o\right)=w\left(a\ , \bar{b}\ \left(a\ \right), o\right). \tag{15}$$

if and only if,

$$\xi\,(x)\,w\left(a,\bar{b}\,(a)\,,o\right)+\delta_{(x,o,\bar{b})}=\xi\,(x)\,w\left(a\ ,\bar{b}\ \ (a\ )\,,o\right)+\delta_{(x,o,\bar{b}\ )}. \tag{16}$$

Thus, $\delta_{(x,o,\bar{b})}=\delta_{(x,o,\bar{b}\ )}$. By continuity, (A.2), the conclusion can be extended to $\mathcal{B}$. Let $\delta_{(x,o,\bar{b})}:=\zeta\,(x)$ for all $\bar{b}\in\mathcal{B}$.

For all $a\in A$ and $\bar{b}\in\mathcal{B}$, define $U\left(a,\bar{b}\,(a)\right)=w\left(a,\bar{b}\,(a)\,,o\right)$ and let $\xi\,(x)$ and $\zeta\,(x)$ denote the multiplicative and additive part of $\phi_{(x,o,\bar{b})}$. Then, for all $x\in\bar{X}$,

$$w\left(a,\bar{b}\,(a)\,,x\right)=\xi\,(x)\,U\left(a,\bar{b}\,(a)\right)+\zeta\,(x)\,,\xi\,(x)>0. \tag{17}$$

This completes the Proof of Lemma 5. ⊓

Let $\hat{\alpha}\,(a,x)=\ \ _{\theta\in}\ \alpha\,(a,x,\theta)$, then Eqs. (6) and (7) imply that for every $x\in\bar{X},\bar{b}\in\mathcal{B}$ and $a\in A$,

$$\xi\,(x)U\left(a,\bar{b}\,(a)\right)+\zeta\,(x)=H\quad\sum_{\theta\in}\lambda\,(a,x,\theta)\,u\left(a,\bar{b}\,(a)\,(\theta)\,,\theta\right)+\hat{\alpha}\,(a,x)\,,a,x\quad. \tag{18}$$

Lemma 6 *The identity (18) holds if and only if* $u\left(a,\bar{b}\,(a)\,(\theta)\,,\theta\right)=u\left(a,\bar{b}\,(a)\,,\theta\ \right)$ *for all* $\theta,\theta\ \in\Theta$, $_{\theta\in}\ \frac{\lambda(a,x,\theta)}{\xi(x)}=\varphi\,(a)\,,\frac{\hat{\alpha}(a,x)}{\xi(x)}=v\,(a)$ *for all* $a\in A$*,*

$$\begin{aligned}&\xi\,(x)\left[\varphi\,(a)\,u\left(a,\bar{b}\,(a)\right)+v\,(a)\right]+\zeta\,(x)\\&=H\quad\sum_{\theta\in}\lambda\,(a,x,\theta)\,u\left(a,\bar{b}\,(a)\,(\theta)\,,\theta\right)+\hat{\alpha}\,(a,x)\,,a,x\quad,\end{aligned} \tag{19}$$

and

$$\sum_{\theta\in}\frac{\lambda\,(a,x,\theta)}{\xi\,(x)}u\left(a,\bar{b}\,(a)\,(\theta)\,,\theta\right)+\frac{\hat{\alpha}\,(a,x)}{\xi\,(x)}=U\left(a,\bar{b}\,(a)\right). \tag{20}$$

Proof (Sufficiency) Let $u\left(a,\bar{b}\,(a)\,(\theta)\,,\theta\right):=u\left(a,\bar{b}\,(a)\right)$ for all $\theta\in\Theta$, $_{\theta\in}\ \frac{\lambda(a,x,\theta)}{\xi(x)}:=\varphi\,(a)$ and $\frac{\hat{\alpha}(a,x)}{\xi(x)}=v\,(a)$ for all $a\in A$ and suppose that (20) holds. Then, Eq. (18) follows from Eq. (19).

(Necessity) Multiply and divide the first argument of H by $\xi\,(x)>0$. Equation (18) may be written as follows:

$$\begin{aligned}&\xi\,(x)\,U\left(a,\bar{b}\,(a)\right)+\zeta\,(x)\\&=H\quad\xi\,(x)\left[\sum_{\theta\in}\frac{\lambda\,(a,x,\theta)}{\xi\,(x)}u\left(a,\bar{b}\,(a)\,(\theta)\,,\theta\right)+\frac{\hat{\alpha}\,(a,x)}{\xi\,(x)}\right],a,x\quad.\end{aligned} \tag{21}$$

Define $V\left(a, \bar{b}(a), x\right) = {}_{\theta \in} \ \frac{\lambda(a,x,\theta)}{\xi(x)} u\left(a, \bar{b}(a)(\theta), \theta\right) + \frac{\hat{\alpha}(a,x)}{\xi(x)}$ then, for every given $(a, x) \in A \times X$ and all $\bar{b}, \bar{b} \ \in \mathcal{B}$,

$$\begin{aligned} &U\left(a, \bar{b}(a)\right) - U\left(a, \bar{b} \ (a)\right) \\ &\quad = \left[H\left(\xi(x) V\left(a, \bar{b}(a), x\right), a, x\right) - H\left(\xi(x) V\left(a, \bar{b} \ (a), x\right), a, x\right)\right] / \xi(x). \end{aligned} \tag{22}$$

Hence, $H(\cdot, a, x)$ is a linear function whose intercept is $\zeta(x)$ and the slope

$$\left[U\left(a, \bar{b}(a)\right) - U\left(a, \bar{b} \ (a)\right)\right] / \left[V\left(a, \bar{b}(a), x\right) - V\left(a, \bar{b} \ (a), x\right)\right] := \kappa(a),$$

is independent of x. Thus,

$$\begin{aligned} &\xi(x) U\left(a, \bar{b}(a)\right) + \zeta(x) \\ &\quad = \kappa(a) \xi(x) \left[\sum_{\theta \in} \frac{\lambda(a, x, \theta)}{\xi(x)} u\left(a, \bar{b}(a)(\theta), \theta\right) + \frac{\hat{\alpha}(a, x)}{\xi(x)}\right] + \zeta(x). \end{aligned} \tag{23}$$

Hence,

$$U\left(a, \bar{b}(a)\right) / \kappa(a) = \sum_{\theta \in} \frac{\lambda(a, x, \theta)}{\xi(x)} u\left(a, \bar{b}(a)(\theta), \theta\right) + \frac{\hat{\alpha}(a, x)}{\xi(x)} \tag{24}$$

is independent of x. However, because $\succcurlyeq_a^x = \succcurlyeq_a^x$ for all a and some $x, x \ \in \bar{X}$, in general, $\lambda(a, x, \theta)$ is not independent of θ. Moreover, because $\hat{\alpha}(a, x) / \xi(x)$ is independent of $\bar{b}$, the first term on the right-hand side of (24) must be independent of x. For this to be true, $u\left(a, \bar{b}(a)(\theta), \theta\right)$ must be independent of θ and ${}_{\theta \in} \ \lambda(a, x, \theta) / \xi(x) := \varphi(a)$ independent of x. Moreover, because the first term on the right-hand side of (24) is independent of x, $\hat{\alpha}(a, x) / \xi(x)$ must also be independent of x. Define $v(a) = \hat{\alpha}(a, x) / \xi(x)$. By definition, $\bar{b}$ is the unique element in its equivalence class that has the property that $u\left(a, \bar{b}(a)(\theta), \theta\right)$ is independent of θ. Define $u\left(a, \bar{b}(a)(\theta), \theta\right) = u\left(a, \bar{b}(a)\right)$ for all $\theta \in$. Hence, $V\left(a, \bar{b}(a), x\right)$ is independent of x, thus $V\left(a, \bar{b}(a), x\right) = \varphi(a) u\left(a, \bar{b}(a)\right) + v(a) = U\left(\bar{b}(a), a\right)$ and, consequently, $\kappa(a) = 1$.

Thus,

$$U\left(a, \bar{b}(a)\right) = \sum_{\theta \in} \frac{\lambda(a, x, \theta)}{\xi(x)} u\left(a, \bar{b}(a)(\theta); \theta\right) + \frac{\hat{\alpha}(a, x)}{\xi(x)}. \tag{25}$$

This completes the Proof of Lemma 6. $\sqcap$

Define $u(a, b(\theta); \theta) = [\varphi(a)\, u(a, b(\theta); \theta) + v(a)]$ then, by Eq. (25),

$$U\left(a, \bar{b}(a)\right) = \sum_{\theta\in} \frac{\lambda(a, x, \theta)}{\xi(x)\,\varphi(a)} \left[\varphi(a)\, u\left(a, \bar{b}(a)(\theta); \theta\right) + v(a)\right]$$
$$= \sum_{\theta\in} \frac{\lambda(a, x, \theta)}{\xi(x)} u\left(a, \bar{b}(a)(\theta); \theta\right). \tag{26}$$

But, by Lemma 6, $_{\theta\in}\ \lambda(a, x, \theta) = \xi(x)\,\varphi(a)$. Hence, the representation (5)implies

$$I \mapsto \sum_{x\in\bar{X}} \left[\sum_{\theta\in} \frac{\lambda\left(a_{I(x)}, x, \theta\right)}{_{\theta\in}\ \lambda\left(a_{I(x)}, x, \theta\right)} u\left(a_{I(x)}, \bar{b}\left(a_{I(x)}\right)(\theta); \theta\right)\right]. \tag{27}$$

For every $(a, b) \in A \times B$, let $\bar{b} \in \mathcal{B}(\succcurlyeq)$ be such that $(a, b) \sim^{x} \left(a, \bar{b}(a)\right)$ for all $x \in \bar{X}$. Then, by the inclusively of $\mathcal{B}$,

$$\sum_{\theta\in} \frac{\lambda\left(a_{I(x)}, x, \theta\right)}{_{\theta\in}\ \lambda\left(a_{I(x)}, x, \theta\right)} u\left(a_{I(x)}, \bar{b}_{I(x)}\left(a_{I(x)}\right)(\theta); \theta\right)$$
$$= \sum_{\theta\in} \frac{\lambda(a, x, \theta)}{_{\theta\in}\ \lambda\left(a_{I(x)}, x, \theta\right)} u\left(a_{I(x)}, b_{I(x)}(\theta); \theta\right). \tag{28}$$

Thus, by the representation (27),

$$I \mapsto \sum_{x\in\bar{X}} \left[\sum_{\theta\in} \frac{\lambda(a, x, \theta)}{_{\theta\in}\ \lambda\left(a_{I(x)}, x, \theta\right)} u\left(a_{I(x)}, b_{I(x)}(\theta); \theta\right)\right].$$

For all $x \in X$, $a \in A$ and $\theta \in \Theta$, define the joint subjective probability distribution on $\times \bar{X}$ by

$$\pi(x, \theta \mid a) = \frac{\lambda(a, x, \theta)}{_{x' \in\bar{X}}\ _{\theta' \in}\ \lambda(a, x', \theta')}. \tag{29}$$

Since $_{\theta\in}\ \lambda(a, x, \theta) = \xi(x)\,\varphi(a)$, for all $x \in \bar{X}$,

$$\sum_{\theta\in} \pi(x, \theta \mid a) = \frac{\xi(x)\,\varphi(a)}{_{x' \in\bar{X}}\ \xi(x')\,\varphi(a)} = \frac{\xi(x)}{_{x' \in\bar{X}}\ \xi(x')}. \tag{30}$$

Define the subjective probability of $x \in \bar{X}$ as follows:

$$\mu(x) = \frac{\xi(x)}{_{x' \in\bar{X}}\ \xi(x')}. \tag{31}$$

Then, the subjective probability of x is given by the marginal distribution on X induced by the joint distributions $\pi\,(\cdot,\cdot \mid a)$ on $X \times$ and is independent of a.

Define the subjective posterior on distribution by

$$\pi\,(\theta \mid x, a) = \frac{\pi\,(x, \theta \mid a)}{\mu\,(x)} = \frac{\lambda\,(a, x, \theta)}{{}_{\theta\in}\ \lambda\,(a, x, \theta)}, \tag{32}$$

and define the subjective prior on by:

$$\pi\,(\theta \mid o, a) = \frac{\lambda\,(a, o, \theta)}{{}_{\theta\in}\ \lambda\,(a, o, \theta)}. \tag{33}$$

Substitute in (27) to obtain the representation (3),

$$I \mapsto \sum_{x\in\bar{X}} \mu\,(x) \left[\sum_{\theta\in} \pi\left(\theta \mid x, a_{I(x)}\right) u\left(a_{I(x),} b_{I(x)}\,(\theta)\,, \theta\right) \right]. \tag{34}$$

(Necessity) The necessity of (A.1), (A.2), and (A.3) follows from Wakker (1989) Theorem III.4.1. To see the necessity of (A.4), suppose that $I_{-x}\,(a, b_{-\theta} r) \succcurlyeq I_{-x}\left(a, b_{-\theta} r\ \right)$, $I_{-x}\left(a, b_{-\theta} r\ \ \right) \succcurlyeq I_{-x}\left(a, b_{-\theta} r\ \ \right)$, and $I_{-x}\ \left(a\ , b_{-\theta} r\ \right) \succcurlyeq I_{-x}$ $\left(a\ , b_{-\theta} r\right)$. For all $(a, b, x) \in A \times B \times \bar{X}$ let $G\,(a, b, x) := \ \ {}_{\theta\ \in\ \ -\{\theta\}}\, \pi\left(\theta\ \ \mid x, a\right) u$ $\left(a, b\left(\theta\ \right), \theta\ \right)$ then, by the representation (3),

$$G\,(a, b, x) + \pi\,(\theta \mid x, a)\, u\,(a, r, \theta) \geq G\left(a, b\ , x\right) + \pi\,(\theta \mid x, a)\, u\left(a, r\ , \theta\right) \tag{35}$$

$$G\left(a, b\ , x\right) + \pi\,(\theta \mid x, a)\, u\left(a, r\ \ , \theta\right) \geq G\,(a, b, x) + \pi\,(\theta \mid x, a)\, u\left(a, r\ \ \ , \theta\right) \tag{36}$$

and

$$G\left(a, b\ \ , x\ \right) + \pi\left(\theta \mid x\ , a\ \right) u\left(a\ , r\ , \theta\right) \geq G\left(a\ , b\ \ \ , x\ \right) + \pi\left(\theta \mid x\ , a\ \right) u\left(a\ , r, \theta\right). \tag{37}$$

But (35) and (36) imply that

$$\begin{aligned} u\left(a, r\ \ , \theta\right) - u\left(a, r\ \ \ , \theta\right) &\geq \frac{G\,(a, b, x) - G\left(a, b\ , x\right)}{\pi\,(\theta \mid x, a)} \\ &\geq u\left(a, r\ , \theta\right) - u\,(a, r, \theta)\,. \end{aligned} \tag{38}$$

Inequality (37) implies

$$\begin{aligned} &u\left(a\ , r\ , \theta\right) - u\left(a\ , r, \theta\right) \\ &\geq \frac{{}_{\theta\ \in\ \ -\{\theta\}}\, \pi\left(\theta\ \ \mid x\ , a\ \right) \left[u\left(a\ , b\ \ \ (\theta\ ), \theta\ \right) - u\left(a\ , b\ \ (\theta\ ), \theta\ \right)\right]}{\pi\,(\theta \mid x\ , a\ )} \end{aligned} \tag{39}$$

But (38) and (39) imply that

$$u\left(a\ ,r\ \ ,\theta\right)-u\left(a\ ,r\ \ ,\theta\right)$$
$$\geq \frac{_{\theta\ \in\ \ -\{\theta\}}\pi\left(\theta\ \ |\ x\ ,a\ \right)\left[u\left(a\ ,b\ \ \ (\theta\ ),\theta\ \right)-u\left(a\ ,b\ \ \ (\theta\ ),\theta\ \right)\right]}{\pi\ (\theta\ |\ x\ ,a\ )}. \tag{40}$$

Hence,

$$\sum_{\theta\ \in\ \ -\{\theta\}}\pi\left(\theta\ \ |\ x\ ,a\ \right)\left[u\left(a\ ,b\ \ (\theta\ ),\theta\ \right)-u\left(a\ ,b\ \ \ (\theta\ ),\theta\ \right)\right]$$
$$+\pi\left(\theta\ |\ x\ ,a\ \right)\left[u\left(a\ ,r\ \ ,\theta\right)-u\left(a\ ,r\ \ ,\theta\right)\right]\geq 0. \tag{41}$$

Thus, $I_{-x}\ \ \left(a\ ,b_{-\theta}r\ \ \right)\succcurlyeq I_{-x}\ \ \left(a\ ,b_{-\theta}r\ \ \right)$.

Let $a\in A$, $I\ \in\mathcal{I}$ and $b,b\ \ \in B$, satisfy $I_{-o}\left(a,b\right)\sim I_{-o}\left(a,b\ \right)$. Then, by (34),

$$\sum_{\theta\in}\pi\left(\theta\mid o,a\right)u\left(a,b\left(\theta\right),\theta\right)=\sum_{\theta\in}\pi\left(\theta\mid o,a\right)u\left(a,b\ \ (\theta),\theta\right) \tag{42}$$

and, by axiom (A.5) and (34)

$$\sum_{x\in X}\frac{\mu\left(x\right)}{1-\mu\left(0\right)}\sum_{\theta\in}\pi\left(\theta\mid x,a\right)u\left(a,b\left(\theta\right),\theta\right)$$
$$=\sum_{x\in X}\frac{\mu\left(x\right)}{1-\mu\left(0\right)}\sum_{\theta\in}\pi\left(\theta\mid x,a\right)u\left(a,b\ \ (\theta),\theta\right). \tag{43}$$

Thus,

$$\sum_{\theta\in}\left[u\left(a,b\left(\theta\right),\theta\right)-u\left(a,b\ \ (\theta),\theta\right)\right]$$
$$\times\left[\pi\left(\theta\mid o,a\right)-\sum_{x\in X}\frac{\mu\left(x\right)}{1-\mu\left(0\right)}\pi\left(\theta\mid x,a\right)\right]=0. \tag{44}$$

This implies that $\pi\left(\theta\mid o,a\right)=\ \ _{x\in X}\ \mu\left(x\right)\pi\left(\theta\mid x,a\right)/\left[1-\mu\left(0\right)\right]$.

(If $\pi\left(\theta\mid o,a\right)>\ \ _{x\in X}\ \mu\left(x\right)\pi\left(\theta\mid x,a\right)/\left[1-\mu\left(0\right)\right]$ for some θ and $\mu\left(o\right)\pi\left(\theta\ \ |\ o,a\right)<\ \ _{x\in X}\ \mu\left(x\right)\pi\left(\theta\ \ |\ x,a\right)/\left[1-\mu\left(0\right)\right]$ for some θ , let $\hat{b},\hat{b}\ \ \in B$ be such that $\hat{b}\left(\theta\right)>b\left(\theta\right)$ and $\hat{b}\ \ \hat{\theta}\Big)=b\ \ \hat{\theta}\Big)$ for all $\hat{\theta}\in\ \ -\{\theta\}$, $\hat{b}\ \ \left(\theta\ \right)>b\ \ \left(\theta\ \right)$ and $\hat{b}\ \ \hat{\theta}\Big)=b\ \ \hat{\theta}\Big)$ for all $\hat{\theta}\in\ \ -\{\theta\ \}$ and $I_{-o}\ \ a,\hat{b}\Big)\sim I_{-o}\ \ a,\hat{b}\ \Big)$. Then,

$$\sum_{\theta\in}\left[u\ \ a,\hat{b}\left(\theta\right),\theta\right)-u\ \ a,\hat{b}\ \ (\theta),\theta\Big)\Big]$$
$$\times\left[\pi\left(\theta\mid o,a\right)-\sum_{x\in X}\frac{\mu\left(x\right)}{1-\mu\left(0\right)}\pi\left(\theta\mid x,a\right)\right]>0. \tag{45}$$

But this contradicts (A.5)). This completes the proof of (a).

(b) Suppose, by way of negation, that there exist continuous, real-valued function $\hat{u}$ *on* $A \times \mathbb{R} \times$ and, for every $a \in A$, there is a joint probability measure $\hat{\pi}(\cdot, \cdot \mid a)$ on $\bar{X} \times \Theta$, distinct from those that figure in the representation (3), such that $\succcurlyeq$ on $\mathcal{I}$ is represented by

$$I \mapsto \sum_{x \in \bar{X}} \hat{\mu}(x) \left[\sum_{\theta \in} \hat{\pi}\left(\theta \mid x, a_{I(x)}\right) \hat{u}\left(a_{I(x)}, b_{I(x)}(\theta), \theta\right) \right], \tag{46}$$

where $\hat{\mu}(x) = {}_{\theta \in} \hat{\pi}(x, \theta \mid a)$ for all $x \in \bar{X}$, and $\hat{\pi}(\theta \mid x, a) = \hat{\pi}(x, \theta \mid a) / \hat{\mu}(x)$ for all $(\theta, x, a) \in \times X \times A$.

Define $\kappa(x) = \hat{\mu}(x) / \mu(x)$, for all $x \in \bar{X}$. Then, the representation (46) may be written as

$$I \mapsto \sum_{x \in \bar{X}} \mu(x) \left[\sum_{\theta \in} \pi\left(\theta \mid x, a_{I(x)}\right) \tilde{\gamma}\left(\theta, x, a_{I(x)}\right) \kappa(x) \hat{u}\left(a_{I(x)}, b_{I(x)}(\theta), \theta\right) \right]. \tag{47}$$

Hence, by (3), $\hat{u}(a, b(\theta), \theta) = u(a, b(\theta), \theta) / \tilde{\gamma}(\theta, x, a) \kappa(x)$. Thus, $\tilde{\gamma}(\theta, x, a)$ $\kappa(x)$ is independent of x. Let $\tilde{\gamma}(\theta, x, a) \kappa(x) = \gamma(\theta, a)$. Then, for $\bar{b} \in \mathcal{B}$,

$$I \mapsto \sum_{x \in \bar{X}} \mu(x) \left[\sum_{\theta \in} \pi\left(\theta \mid x, a_{I(x)}\right) \frac{u\left(a, \bar{b}(a)\right)}{\gamma(\theta, a)} \right]. \tag{48}$$

Let $\hat{b} \in B$ be defined by $u \;\; a, \hat{b}(\theta), \theta\Big) = u\left(a, \bar{b}(a)\right) / \gamma(\theta, a)$ for all $\theta \in$ and $a \in A$. Then, $\hat{b} \sim_a^x \bar{b}(a)$ for all $x \in \bar{X}$, and, by Definition 1, $\hat{b} \in \mathcal{B}$. Moreover, if $\gamma(\cdot, \cdot)$ is not a constant function, then $\hat{b} = \bar{b}$. This contradicts the uniqueness of $\bar{b}$ in Definition 1. Thus, $\gamma(\theta, a) = \bar{\gamma}$ for all $\theta \in$ and $a \in A$. But

$$1 = \sum_{x \in \bar{X}} \sum_{\theta \in} \hat{\pi}\left(\theta, x \mid a_{I(x)}\right) = \bar{\gamma} \sum_{x \in \bar{X}} \sum_{\theta \in} \pi(\theta, x \mid a) = \bar{\gamma}. \tag{49}$$

Hence, $\hat{\pi}(\theta, x \mid a) = \pi(\theta, x \mid a)$ for all $(\theta, x) \in \times \bar{X}$ and $a \in A$.

Next, consider the uniqueness of the utility functions. Clearly, if $\hat{u}(a, \cdot, \theta) = mu(a, \cdot, \theta) + k, m > 0$, for all $a \in A$ and $\theta \in \Theta$, then

$$\sum_{x \in \bar{X}} \sum_{\theta \in} \pi(\theta \mid x, a) \hat{u}(a, b(\theta), \theta) = m \sum_{x \in \bar{X}} \sum_{\theta \in} \pi(\theta, x \mid a) u(a, b(\theta), \theta) + k, \tag{50}$$

and $\{\hat{u}(a, \cdot, \theta) \mid a \in A, \theta \in \;\; \}$ is another utility function that, jointly with $\{\pi(\cdot, \cdot \mid a)\}_{a \in A}$ represents $\succcurlyeq$.

Suppose that $\hat{u}(a, \cdot, \theta) = m(a, \theta) u(a, \cdot, \theta) + k$, where $m(\cdot, \cdot)$ is not a constant function. Define $\hat{b}(\theta, a)$ by $m(a, \theta) u\ \ a, \hat{b}(\theta, a), \theta\big) = u\left(a, \bar{b}(a)\right)$ for all $\theta \in$ and $a \ \in A$. That such $\hat{b}$ exists follows from the exclusivity of $\mathcal{B}$. By definition, $\hat{b}(\cdot, a) \sim_a^x \bar{b}(a)$ for all x and $\hat{b} = \bar{b}$. This contradicts the uniqueness of $\bar{b}$ in Definition 1. Hence, $m(a, \theta)$ must be a constant function.

Consider next $\hat{u}(a, \cdot, \theta) = mu(a, \cdot, \theta) + k(\theta, a)$, and suppose that $k(\cdot, a)$ is not a constant function. Let $\bar{k}(x, a) = \ \ {}_{\theta \in}\ \pi(\theta \mid x, a) k(\theta, a)$. Take $a, a \ \in A$ and $\bar{b}, \bar{b} \ \in \mathcal{B}$ such that $I_{-x}\ \left(a, \bar{b}(a)\right) \sim I_{-x}\ \left(a\ , \bar{b}\left(a\ \right)\right)$ and $\left[\bar{k}(x, a) - \bar{k}\left(x, a\ \right)\right] = 0$ for some strategies I, I and observation x. Let

$$J = \sum_{x \in \bar{X} - \{x\ \}} \sum_{\theta \in} \left[\pi\left(\theta, x \mid a_{I(x)}\right) u\left(a_{I(x)}, b_{I(x)}(\theta), \theta\right) - \pi\left(\theta \mid x, a_{I\ (x)}\right) u\left(a_{I\ (x)}, b_{I\ (x)}(\theta), \theta\right)\right]$$

and

$$\hat{J} = \sum_{x \in \bar{X} - \{x\ \}} \sum_{\theta \in} \left[\pi\left(\theta, x \mid a_{I(x)}\right) \hat{u}\left(a_{I(x)}, b_{I(x)}(\theta), \theta\right) - \pi\left(\theta \mid x, a_{I\ (x)}\right) \hat{u}\left(a_{I\ (x)}, b_{I\ (x)}(\theta), \theta\right)\right].$$

Then,

$$\hat{u}\left(a, \bar{b}(a)\right) - \hat{u}\left(a\ , \bar{b}\left(a\ \right)\right) + \hat{J} = m\left[u\left(a, \bar{b}(a)\right) - u\left(a\ , \bar{b}\left(a\ \right)\right) + J\right] + \left[\bar{k}(x, a) - \bar{k}\left(x, a\ \right)\right]. \tag{51}$$

But $u\left(a, \bar{b}(a)\right) - u\left(a\ , \bar{b}\left(a\ \right)\right) + J = 0$ and, by Eq. (51) $\hat{u}\left(a, \bar{b}(a)\right) - \hat{u}\left(a\ , \bar{b}\left(a\ \right)\right) + \hat{J} = 0$. Hence, $\hat{u}(\cdot, \theta)$ does not represent $\succcurlyeq$. This completes the proof of (b).

(c) Next, I show that if $\bar{b} \in B^A$ satisfies $u\left(a, \bar{b}(a)(\theta), \theta\right) = u\left(a, \bar{b}(a)\left(\theta\ \right), \theta\ \right)$ for all $\theta, \theta\ , \in \Theta$, $a \in A$ then $\bar{b} \in \mathcal{B}$. Let $u\left(a, \bar{b}(a)(\theta), \theta\right) = g\left(a, \bar{b}(a)\right)$. Suppose that representation (3) holds and let $I, I\ , I\ \ , I\ \ \ \in \mathcal{I}$, $a, a\ , a\ \ , a\ \ \ \in A$ and $x, x\ \in \bar{X}$, such that $I_{-x}\left(a, \bar{b}(a)\right) \sim I_{-x}\left(a\ , \bar{b}\left(a\ \right)\right)$, $I_{-x}\left(a\ \ , \bar{b}\left(a\ \ \right)\right) \sim I_{-x}\left(a\ \ \ , \bar{b}\left(a\ \ \ \right)\right)$ and $I_{-x}\ \left(a\ , \bar{b}\left(a\ \right)\right) \sim I_{-x}\ \left(a, \bar{b}(a)\right)$. Then, the representation (5) implies that

$$\sum_{\hat{x} \in \bar{X} - \{x\}} w\ \ a_{I(\hat{x})}, b_{I(\hat{x})}, \hat{x}\big) + \mu(x) g\left(a, \bar{b}(a)\right) = \sum_{\hat{x} \in \bar{X} - \{x\}} w\ \ a_{I\ (\hat{x})}, b_{I\ (\hat{x})}, \hat{x}\big) + \mu(x) g\left(a\ , \bar{b}\left(a\ \right)\right) \tag{52}$$

$$\sum_{\hat{x} \in \bar{X} - \{x\}} w\ \ a_{I\ (\hat{x})}, b_{I\ (\hat{x})}, \hat{x}\big) + \mu(x) g\left(a\ \ , \bar{b}\left(a\ \ \right)\right) = \sum_{\hat{x} \in \bar{X} - \{x\}} w\ \ a_{I(\hat{x})}, b_{I(\hat{x})}, \hat{x}\big) + \mu(x) g\left(a\ \ \ , \bar{b}\left(a\ \ \ \right)\right) \tag{53}$$

and

$$\sum_{\hat{x}\in\bar{X}-\{x\ \}} w\ \ a_{I\ (\hat{x})}, b_{I\ (\hat{x})}, \hat{x}\Big) + \mu\left(x\ \right) g\left(a\ , \bar{b}\left(a\ \right)\right)$$
$$= \sum_{\hat{x}\in\bar{X}-\{x\ \}} w\ \ a_{I\ (\hat{x})}, b_{I\ (\hat{x})}, \hat{x}\Big) + \mu\left(x\ \right) g\left(a, \bar{b}\left(a\right)\right). \tag{54}$$

Equations (52) and (53) imply that

$$g\left(a, \bar{b}\left(a\right)\right) - g\left(a\ , \bar{b}\left(a\ \right)\right) = g\left(a\ \ , \bar{b}\left(a\ \ \right)\right) - g\left(a\ \ \ , \bar{b}\left(a\ \ \ \right)\right). \tag{55}$$

Equality (54) implies

$$\frac{{}_{\hat{x}\in\bar{X}-\{x\ \}}\left[w\ \ a_{I\ (\hat{x})}, b_{I\ (\hat{x})}, \hat{x}\Big) - w\ \ a_{I\ (\hat{x})}, b_{I\ (\hat{x})}, \hat{x}\Big)\right]}{\mu\left(x\ \right)}$$
$$= g\left(a, \bar{b}\left(a\right)\right) - g\left(a\ , \bar{b}\left(a\ \right)\right). \tag{56}$$

Thus,

$$\sum_{\hat{x}\in\bar{X}-\{x\ \}} w\ \ a_{I\ (\hat{x})}, b_{I\ (\hat{x})}, \hat{x}\Big) + g\left(a\ \ \ , \bar{b}\left(a\ \ \ \right)\right)$$
$$= \sum_{\hat{x}\in\bar{X}-\{x\ \}} w\ \ a_{I\ (\hat{x})}, b_{I\ (\hat{x})}, \hat{x}\Big) + g\left(a\ \ , \bar{b}\left(a\ \ \right)\right) \tag{57}$$

Hence, $I_{-x}\ \left(a\ \ , \bar{b}\left(a\ \ \right)\right) \sim I_{-x}\ \left(a\ \ \ , \bar{b}\left(a\ \ \ \right)\right)$ and $\bar{b} \in \mathcal{B}$.

To show the necessity of (A.5), let $a \in A$, $I \in \mathcal{I}$ and $b, b\ \in B$, by the representation $I_{-o}\left(a, b\right) \sim I_{-o}\left(a, b\ \right)$ if and only if

$$\sum_{\theta\in} \pi\left(\theta \mid o, a\right) u\left(a, b\left(\theta\right), \theta\right) = \sum_{\theta\in} \pi\left(\theta \mid o, a\right) u\left(a, b\ \ \left(\theta\right), \theta\right). \tag{58}$$

But $\pi\left(\theta \mid o, a\right) = \quad {}_{x\in X}\, \mu\left(x\right) \pi\left(\theta \mid x, a\right) / \left[1 - \mu\left(0\right)\right]$. Thus, (58) holds if and only if

$$\sum_{x\in X} \mu\left(x\right) \sum_{\theta\in} \pi\left(\theta \mid X, a\right) u\left(a, b\left(\theta\right), \theta\right)$$
$$= \sum_{x\in X} \mu\left(x\right) \sum_{\theta\in} \pi\left(\theta \mid x, a\right) u\left(a, b\ \ \left(\theta\right), \theta\right). \tag{59}$$

But (59) is valid if and only if $I^{-o}\left(a, b\right) \sim I^{-o}\left(a, b\ \right)$.

For all I and x,, let $K(I,x) = {}_{y\in X-\{x\}}\, \mu(y) \quad {}_{\theta\in}\ \pi(\theta \mid x,a)\, u\left(a_{I(y)}\right.$, $\left.b_{I(y)}(\theta), \theta\right)$. To show the necessity of (A.6), then $I_{-x}\left(a, \bar{b}(a)\right) \succcurlyeq I_{-x}$ $\left(a\ , \bar{b}\left(a\ \right)\right)$ if and only if

$$K(I,x) + u\left(a, \bar{b}(a)\right) \geq K(I\ ,x) + u\left(a\ , \bar{b}\left(a\ \right)\right) \tag{60}$$

if and only if

$$K(I,x) + u\left(a, \bar{b}\ \ (a)\right) \geq K(I\ ,x) + u\left(a\ , \bar{b}\ \ \left(a\ \right)\right) \tag{61}$$

if and only if $I_{-x}\left(a, \bar{b}\ \ (a)\right) \succcurlyeq I_{-x}\left(a\ , \bar{b}\ \ \left(a\ \right)\right)$.

References

Anscombe, F., Aumann, R.: A definition of subjective probability. Ann Math Stat **34**, 199–205 (1963)

Karni, E.: Subjective expected utility without states of the world. J Math Econ **42**, 325–342 (2006)

Karni, E.: A theory of Bayesian decision making. Econ Theory **48**, 125–146 (2011)

Karni, E.: Subjective probabilities on a state-space. Am Econ J Microecon **3**, 172–185 (2011)

Kreps, D.M., Porteus, E.L.: Temporal resolution of uncertainty and dynamic choice theory. Econometrica **46**, 185–200 (1978)

Kreps, D.M., Porteus, E.L.: Temporal von Neumann–Morgestern and induced preferences. J Econ Theory **20**, 81–109 (1979)

Machina, M.: Temporal risk and the nature of induced preferences. J Econ Theory **33**, 199–231 (1984)

Mirrlees, J.: Notes on welfare economics, information and uncertainty. In: Balch, M., McFadden, D., Wu, S. (eds.) Essays in Economic Behavior Under Uncertainty, Amsterdam: North Holland (1974)

Mirrlees, J.: The optimal structure of authority and incentives within an organization. Bell J Econ **7**, 105–113 (1976)

Nau, R.: Coherent decision analysis with inseparable probabilities and utilities. J Risk Uncertain **10**, 71–91 (1995)

Nau, R.: Risk, ambiguity, and state-preference theory. Econ Theory **48**, 437–467 (2011)

Savage, L.J.: The Foundations of Statistics. New York: Wiley (1954)

Wakker, P.P.: Subjective probabilities for state dependent continuous utility. Math Soc Sci **14**, 289–298 (1987)

Wakker, P.P.: Additive Representations of Preferences. Dordrecht: Kluwer Academic Publishers (1989)